Int. J. Multiphase Flow Vol. 9, No. 3, pp. 227-250, 1983
Printed in Great Britain.
0301-9322/83/030227-24$03.00/0
Pergamon/Elsevier

# ON THE SLOW TRANSLATION OF A SOLID SUBMERGED IN A FLUID WITH A SURFACTANT SURFACE FILM—II

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(*Received* 29 *June* 1982; *in revised form* 1 *October* 1982)

**Abstract**—In Shail & Gooden (1982) the problem of a solid particle translating in a semi-infinite fluid, whose surface is contaminated with a surfactant film, was examined in the quasi-steady Stokes flow régime. Various linearised models governing the variation of film concentration were considered, but the analysis was approximate in that the fluid motion generated was represented by that due to a Stokeslet situated at the centre of the particle. In this paper we remove the latter restriction and treat two specific solids, namely a rigid flat circular disk and a sphere, which move axisymmetrically perpendicular to the fluid surface. This surface is assumed to remain plane throughout the motion. The velocity field in the translating-disk problem is represented in terms of harmonic functions, and the resulting mixed boundary-value problems are reduced, for each of the film behaviours examined, to the solution of sets of simultaneous Fredholm integral equations of the second kind. These equations are solved both iteratively and numerically, and the drag on the disk is computed. For the sphere a stream-function formulation in bispherical coordinates is used. Application of the boundary conditions at the sphere and film results in infinite sets of simultaneous linear equations for the coefficients in the eigenfunction expansion of the stream function. These equations are solved by the method of truncation, and the drag on the sphere is determined.

## 1. INTRODUCTION

In the first paper in this series (Shail & Gooden 1982), hereafter referred to as I, a start was made on examining a class of problems in which a solid particle translates in a semi-infinite fluid whose surface is contaminated with a monomolecular surfactant film. The fluid motion is assumed to be slow, quasi-steady and axisymmetric, permitting use of the linearised time-independent Stokes equations of motion. The constitutive properties of the surfactant film are described in terms of the Boussinesq coefficients of surface shear and dilatational viscosity, $\eta$ and $\kappa$. Further, the dynamic boundary conditions on the fluid velocity components in the surface film are those due to Scriven (1960). These conditions (equations [1] and [2] of I) involve the surface pressure $p_s$ (or, equivalently, the surface concentration $n$ of surfactant molecules via the equation of state of the film), which in general will vary with position in the film. Thus, in I various physical processes governing the variation of surfactant concentration were considered. These comprise surface diffusion, adsorbtion and desorbtion, and for a soluble surfactant bulk diffusion into the film from the substrate fluid. In all cases it is necessary to linearise about an equilibrium concentration in the manner suggested by Levich (1962) in order to obtain tractable boundary conditions. The further assumption that the surfactant-covered fluid boundary remains plane during the motion is also made.

In I the various models outlined above were applied to the problem of determining the fluid motion due to an axial Stokeslet in the bulk fluid. This singular solution was then used, in conjunction with arguments due to Brenner (1962), to derive approximate expressions for the drag on an arbitrary axisymmetric body which translates in a direction perpendicular to the surfactant film. In general these drag formulae are correct to $O(a^2/h^2)$, where $a$ is a typical dimension of the solid and $h$ measures its depth below the film, and the analysis gives rise to no fewer than five dimensionless groups. However, being valid only when $a/h \ll 1$, these formulae are of limited value in assessing the influence of the surfactant film on the resistive force experienced by the translating solid.

It is the purpose of this paper to remove, for two distinct geometries, the restriction $a/h \ll 1$. In particular for the various film processes we formulate and solve the appropriate axisymmetric boundary-value problems which arise in the cases of a translating disk and a sphere. The disk moves normal to its plane, which remains parallel to the surfactant film, and the sphere moves without rotation in a direction normal to the film.

In section 2 the basic equations and boundary conditions of the theory are recapitulated, and in section 3 the disk problem is formulated using a representation of the bulk-fluid velocity field in terms of harmonic functions. The boundary conditions at the surfactant film and on the plane of the disk then lead to sets of dual integral equations, which are reduced using well-known methods (see, for example, Sneddon 1966) to the solution of coupled sets of Fredholm integral equations of the second kind. In the cases of surface diffusion and adsorption/desorption, these integral equations are integrated both iteratively (when $a/h \ll 1$) and numerically in section 4, and representative numerical results are presented which exhibit the effects of the surface film on the drag force on the disk. For bulk diffusion into the film from the substrate our linearised model only encompasses the case of small Peclet number in which diffusion dominates over convection. As an example, in section 3 the governing set of three simultaneous Fredholm integral equations is derived for the situation of a sealed disk, i.e. the normal derivative of the solute concentration vanishes on the disk. Since this class of problem does not seem to be of as much interest as the surface diffusion and adsorption/desorption cases, we restrict ourselves to some remarks in section 4 concerning the iterative solution of the example presented.

Section 5 deals with the analogous boundary-value problems which arise when the translating body is a sphere. Attention is focussed on the surface diffusion and adsorption/desorption film processes, and a representation of the velocity field in terms of a stream function is used.† Bispherical coordinates and the eigenfunction expansion of the stream function due to Stimson & Jeffery (1926) are appropriate, and the film boundary conditions and the no-slip condition on the sphere lead to infinite sets of simultaneous linear equations for the coefficients in the stream-function expansion. These sets of equations are truncated and solved numerically in section 6, and the drag force computed for a range of values of the ratio sphere radius to depth of centre below the film, and the dimensionless groups introduced in I.

## 2. BASIC EQUATIONS

A semi-infinite expanse of viscous incompressible fluid, with coefficient of viscosity $\mu$, occupies the region $z > 0$, where $(\rho, \phi, z)$ are cylindrical polar coordinates with $z$-axis vertically downwards. The surface $z = 0$ is contaminated with a monomolecular surfactant film whose coefficients of surface shear and dilational viscosity are $\eta$ and $\kappa$, respectively. The motion of the bulk fluid is caused by the steady translation, with constant speed $U\mathbf{z}$, of a solid body whose surface $S$ has the $z$-axis as an axis of rotational symmetry. Thus, the fluid velocity vector $\mathbf{v}$ is axisymmetric with cylindrical components $(u(\rho, z), 0, w(\rho, z))$, the motion being assumed to be sufficiently slow for the quasi-steady Stokes creeping-motion approximation to be made. This requires that $Ua/\nu \ll 1$ and $Ua^2/\nu h \ll 1$, where $\nu$ is the kinematic viscosity of the bulk fluid, $a$ is a typical dimension of the translating solid, and $h$ a measure of its depth below $z = 0$. The linearised time-independent equations of motion and continuity are then

$$\frac{\partial p}{\partial \rho} = \mu\left(\nabla^2 u - \frac{u}{\rho^2}\right), \qquad [1]$$

†It is clear that the disk problems of section 3 can also be treated using a stream function. However a formulation in terms of harmonic functions is preferred since this approach is also available for non-axially symmetric configurations.

$$\frac{\partial p}{\partial z} = \mu \nabla^2 w, \qquad [2]$$

and

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho u) + \frac{\partial w}{\partial z} = 0, \qquad [3]$$

where

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2},$$

and $p$ is the dynamic pressure field.

At the boundary $S$ of the solid the no-slip condition is imposed, i.e.

$$\mathbf{v}(\rho, z) = U\mathbf{z}, \quad (\rho, z) \in S; \qquad [4]$$

further $\mathbf{v}$, $p$, and all components of stress in the fluid are required to tend to zero as $\rho^2 + z^2 \to \infty$. We also suppose as in I that the surfactant layer remains plane and incident with $z = 0$ throughout the motion, which implies that

$$w = 0 \text{ on } z = 0, 0 \leq \rho < \infty. \qquad [5]$$

The remaining boundary conditions applied at $z = 0$ depend on the physics of the film process which maintains the concentration of surfactant molecules in the film, and these conditions are extensively discussed in I. For the case of isothermal surface diffusion in a gaseous film we have that

$$-\mu\frac{\partial u}{\partial z} = -\alpha u + (\eta + \kappa)\left\{\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial u}{\partial \rho}\right) - \frac{u}{\rho^2}\right\} \text{ on } z = 0, \qquad [6]$$

where $\alpha = kTn_0/D_s$. Here $k$ is Boltzmann's constant, $T$ the temperature, $n_0$ the equilibrium concentration and $D_s$ the coefficient of surface diffusion. When the dominating film process is adsorption from and desorption to the bulk fluid of surfactant, the appropriate linearised boundary condition is

$$-\mu\frac{\partial u}{\partial z} = \left(\eta + \kappa + \frac{n_0 kT}{\beta}\right)\left\{\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial u}{\partial \rho}\right) - \frac{u}{\rho^2}\right\} \text{ on } z = 0, \qquad [7]$$

where the constant $\beta^{-1}$ is a measure of the time required for the establishment of an adsorbtion equilibrium (Levich 1962). It is worth noting that using the equation of continuity [3], the terms in braces in [6] and [7] can be replaced by $-\partial^2 w/\partial\rho\partial z$.

Consider next bulk diffusion from the substrate into the film. For a soluble surfactant the solute concentration $c(\rho, z)$ is defined throughout the bulk fluid, and we write $c = c_0 + c'$, where $c_0$ is the constant equilibrium concentration. When the Peclet number $Ua/D_0 \ll 1$, where $D_0$ is the bulk diffusion coefficient, $c'$ satisfies Laplace's equation

$$\nabla^2 c' = 0. \qquad [8]$$

According to I the equation of surface mass balance in the film gives the linearised condition

$$\frac{n_0}{\rho}\frac{\partial}{\partial \rho}(\rho u) = \frac{D_0}{h_0}\frac{\partial n'}{\partial z} \text{ on } z = 0, \qquad [9]$$

where $n' = h_0 c'$, $h_0$ being the adsorbtion depth. Equation [9] must be taken in conjunction with the Scriven boundary condition

$$-\mu\frac{\partial u}{\partial z} = -\frac{\partial p_s}{\partial \rho} + (\eta + \kappa)\left\{\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial u}{\partial \rho}\right) - \frac{u}{\rho^2}\right\} \text{ on } z = 0, \qquad [10]$$

and an equation of state, which for a gaseous film is

$$p_s = kTn. \qquad [11]$$

Note that the surface concentration $n(\rho)$ is given by $n(\rho) = n_0 + n'(\rho, 0)$, since $n'(\rho, z)$, the fictitious "surface concentration" is defined throughout the fluid by $n' = h_0 c'$.

## 3. THE TRANSLATING DISK PROBLEM

The equations and model conditions of section 2 are now applied to the particular case in which the translating body is a thin rigid circular disk of radius $a$, at an instantaneous depth $h$ below the surface. The no-slip requirements [4] now read

$$u(\rho, h) = 0, \qquad [12]$$

$$w(\rho, h) = U, \text{ both on } z = h, 0 \le \rho \le a. \qquad [13]$$

Further the pressure $p$ and all components of the stress tensor must be continuous across $z = h$ for $a \le \rho < \infty$.

In order to construct a representation of the velocity field $\mathbf{v}$, we observe that for the problem of the steady broadside motion of a disk through an unbounded fluid at rest at infinity, the velocity and pressure fields, $\mathbf{v}_0$ and $p_0$, can be written as

$$\mathbf{v}_0 = (z - h)\nabla\psi - \psi\mathbf{z}, \qquad [14]$$

$$p_0 = 2\mu\frac{\partial \psi}{\partial z}, \qquad [15]$$

where $\psi$ is an axisymmetric harmonic function and the disk lies in the plane $z = h$ (Gupta 1957). Assuming that the pressure at infinity is zero, a suitable form for $\psi$, even with respect to the plane $z = h$, is

$$\psi = \int_0^\infty A(s)J_0(s\rho)\,e^{-s|z-h|}\,ds, \qquad [16]$$

where $A(s)$ is determined from the dual integral equations arising from the boundary conditions on $z = h$. Note that [14] gives $u_0(\rho, h) = 0$ for all $\rho$. For our semi-infinite fluid problem we next add to [14] a regular solution $\mathbf{v}_1$ of the Navier–Stokes equations, to be chosen so as to satisfy [5] and the appropriate condition from [6], [7], or [9] and [10]. A suitable form in terms of harmonic functions $V$, $W$ is

$$\mathbf{v}_1 = (z-h)\nabla V - V\mathbf{z} + \nabla W, \tag{17}$$

$$p_1 = 2\mu \frac{\partial V}{\partial z}, \tag{18}$$

where

$$V = \int_0^\infty C(s) J_0(s\rho)\, e^{-s(z-h)}\, ds, \tag{19}$$

$$W = \int_0^\infty D(s) J_0(s\rho)\, e^{-s(z-h)}\, ds. \tag{20}$$

The velocity field $\mathbf{v}_0 + \mathbf{v}_1$ cannot be made to satisfy both [12] and [13] on the disk, but only [13], say. Thus we add to $\mathbf{v}_0 + \mathbf{v}_1$ a further solution $\mathbf{v}_2$, chosen to enable satisfaction of [12], whilst giving a zero contribution to the $z$-component of velocity on the disk. A suitable solution in terms of harmonics $\psi'$ and $\chi$ is

$$\mathbf{v}_2 = (z-h)\nabla\psi' - \psi'\mathbf{z} + \nabla\chi, \tag{21}$$

$$p_2 = 2\mu \frac{\partial \psi'}{\partial z}, \tag{22}$$

where $\psi' = \partial\chi/\partial z$ is odd with respect to $z = h$, thereby ensuring that $u_2(\rho, h) = 0$ for $\rho > a$. Thus for $\psi'$ and $\chi$ we have the forms

$$\psi'(\rho, z) = \pm \int_0^\infty B(s) J_0(s\rho)\, e^{-s|z-h|}\, ds, \tag{23}$$

$$\chi(\rho, s) = \int_0^\infty s^{-1} B(s) J_0(s\rho)\, e^{-s|z-h|}\, ds, \tag{24}$$

where the positive sign is taken in [23] for $z - h < 0$ and the negative for $z - h > 0$. Combining together the various solutions the complete velocity and pressure fields are

$$\begin{aligned}\mathbf{v} &= \mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2 \\ &= (z-h)\nabla(\psi + \psi' + V) - (\psi + \psi' + V)\mathbf{z} + \nabla(W + \chi),\end{aligned} \tag{25}$$

$$p = 2\mu \frac{\partial}{\partial z}(\psi + \psi' + V). \tag{26}$$

Further, the stress components $\tau_{\rho z}$, $\tau_{zz}$ in the bulk fluid are found as

$$\tau_{\rho z} = 2\mu\left\{(z-h)\frac{\partial^2}{\partial\rho\,\partial z}(\psi + \psi' + V) + \frac{\partial\psi'}{\partial\rho} + \frac{\partial^2 W}{\partial\rho\,\partial z}\right\}, \tag{27}$$

and

$$\tau_{zz} = 2\mu\left\{(z-h)\frac{\partial^2}{\partial z^2}(\psi + \psi' + V) - \frac{\partial}{\partial z}(\psi + V) + \frac{\partial^2 W}{\partial z^2}\right\}. \tag{28}$$

### 3.1 *Surface diffusion in the film*

Suppose that the dominating film process is surface diffusion. Then the solution [25] and [26] must satisfy conditions [5] and [6] on $z = 0$, [12] and [13] on the disk, and the continuity requirements for $p$ and the stresses across the plane $z = h$ for $\rho > a$. Using the quoted Hankel representations of the various harmonic functions, [5] and [6] give

$$sh\{-(A+B)\mathrm{e}^{-sh} + C\,\mathrm{e}^{sh}\} - A\,\mathrm{e}^{-sh} - (C+sD)\,\mathrm{e}^{sh} = 0. \qquad [29]$$

and

$$\mu[sh\{-(A+B)\mathrm{e}^{-sh} + C\,\mathrm{e}^{sh}\} + (A+2B)\mathrm{e}^{-sh} + (C-sD)\,\mathrm{e}^{sh}] \\ = -\{\alpha + s^2(\eta+\kappa)\}[h\{(A+B)\mathrm{e}^{-sh} + C\,\mathrm{e}^{sh}\} - s^{-1}B\,\mathrm{e}^{-sh} - D\,\mathrm{e}^{sh}]. \qquad [30]$$

Conditions [12] and [13] in the plane of the disk require that for $0 \le \rho \le a$,

$$\int_0^\infty \{B(s) + sD(s)\}J_1(s\rho)\,\mathrm{d}s = 0, \qquad [31]$$

and

$$\int_0^\infty \{A(s) + C(s) + sD(s)\}J_0(s\rho)\,\mathrm{d}s = -U. \qquad [32]$$

From [16] and [26] continuity of pressure across $z = h$ for $\rho > a$ implies that

$$\int_0^\infty sA(s)J_0(s\rho)\,\mathrm{d}s = 0,\ \rho > a, \qquad [33]$$

and continuity of the stress components [27] and [28] requires the continuity of $\partial\psi'/\partial\rho$ across $z = h$ for $\rho > a$, i.e.

$$\int_0^\infty B(s)J_1(s\rho)\,\mathrm{d}s = 0,\ \rho > a. \qquad [34]$$

Equations [29] and [30] can be solved to express $C$ and $D$ in terms of $A$ and $B$; [31] to [34] then furnish a set of coupled dual integral equations for $A$ and $B$, which may be solved using standard techniques (Sneddon 1966). To this end we introduce functions $g(t)$, $j(t)$ such that

$$A(s) = \int_0^a g(t)\cos st\,\mathrm{d}t, \qquad [35]$$

and

$$B(s) = -\int_0^a t^{-1}j(t)\sin st\,\mathrm{d}t; \qquad [36]$$

[33] and [34] are thereby satisfied identically. Also, substituting [35] into [32], interchanging orders of integration and using the result

$$\int_0^\infty J_0(s\rho)\cos st\,\mathrm{d}s = (\rho^2 - t^2)^{-\frac{1}{2}}H(\rho - t),$$

we obtain

$$\int_0^\rho \frac{g(t)}{(\rho^2 - t^2)^{\frac{1}{2}}} \mathrm{d}t = -U - \int_0^\infty \{C(s) + sD(s)\} J_0(s\rho)\, \mathrm{d}s, 0 \le \rho \le a,$$

an Abel integral equation for $g(t)$ whose solution may be reduced to

$$g(t) = -\frac{2U}{\pi} - \frac{2}{\pi}\int_0^\infty \{C(s) + sD(s)\} \cos st \, \mathrm{d}s, 0 \le t \le a. \qquad [37]$$

A similar calculation using [36] and [31] leads to

$$j(t) = \frac{2t}{\pi}\int_0^\infty sD(s) \sin st \, \mathrm{d}s, 0 \le t \le a. \qquad [38]$$

We next solve [29] and [30] for $C$, $D$ in terms of $A$, $B$ and thence, using [35] and [36], express $C$ and $D$ in terms of $g(t)$ and $j(t)$. Substituting for $C$ and $D$ in [37] and [38] then supplies the following pair of coupled Fredholm integral equations of the second kind for $g(t)$ and $j(t)$:

$$g(t) - \frac{2}{\pi}\int_0^a g(u)\left\{\int_0^\infty \frac{q(1 + 2sh + 2s^2h^2) + 2\mu s(1 + 2sh)}{q + 2\mu s} \mathrm{e}^{-2sh} \cos su \cos st \, \mathrm{d}s\right\} \mathrm{d}u$$
$$= -\frac{2U}{\pi} - \frac{4}{\pi}\int_0^a u^{-1} j(u)\left\{\int_0^\infty \frac{s^2h(qh + 2\mu)}{q + 2\mu s} \mathrm{e}^{-2sh} \sin su \cos st \, \mathrm{d}s\right\} \mathrm{d}u, \qquad [39]$$

and

$$j(t) - \frac{2t}{\pi}\int_0^a u^{-1} j(u)\left\{\int_0^\infty \frac{q(1 - 2sh + 2s^2h^2) - 2\mu s(1 - 2sh)}{q + 2\mu s} \mathrm{e}^{-2sh} \sin su \sin st \, \mathrm{d}s\right\} \mathrm{d}u$$
$$= -\frac{4t}{\pi}\int_0^a g(u)\left\{\int_0^\infty \frac{s^2h(qh + 2\mu)}{q + 2\mu s} \mathrm{e}^{-2sh} \cos su \sin st \, \mathrm{d}s\right\} \mathrm{d}u, \qquad [40]$$

where $q = \alpha + s^2(\eta + \kappa)$ and $0 \le t \le a$. Equations [39] and [40] can be conveniently non-dimensionalised by writing $x = t/a$, $y = u/a$, $\epsilon = a/h$, $G(x) = U^{-1}g(t)$, $J(x) = U^{-1}t^{-1}j(t)$, and $p = sh$, giving

$$G(x) - \int_0^1 K_1(x, y) G(y)\, \mathrm{d}y = -\frac{2}{\pi} - \int_0^1 K_3(x, y) J(y)\, \mathrm{d}y, \qquad [41]$$

and

$$J(x) - \int_0^1 K_2(x, y) J(y)\, \mathrm{d}y = -\int_0^1 K_3(y, x) G(y)\, \mathrm{d}y, 0 \le x \le 1, \qquad [42]$$

where the kernels $K_i$, $i = 1, \ldots, 3$ are defined by

$$K_1(x, y) = \frac{2\epsilon}{\pi}\int_0^\infty \left\{1 + 2p + 2p^2 - \frac{2\epsilon N_1 p^3}{1 + \epsilon N_1 p + \epsilon^2 N_2 p^2}\right\} \mathrm{e}^{-2p} \cos \epsilon px \cos \epsilon py \, \mathrm{d}p, \qquad [43]$$

$$K_2(x, y) = \frac{2\epsilon}{\pi}\int_0^\infty \left\{1 - 2p + 2p^2 - \frac{2\epsilon N_1 p(1 - p)^2}{1 + \epsilon N_1 p + \epsilon^2 N_2 p^2}\right\} \mathrm{e}^{-2p} \sin \epsilon px \sin \epsilon py \, \mathrm{d}p, \qquad [44]$$

and

$$K_3(x, y) = \frac{4\epsilon}{\pi} \int_0^\infty \left\{ p^2 + \frac{\epsilon N_1 p^2}{1 + \epsilon N_1 p + \epsilon^2 N_2 p^2} \right\} e^{-2p} \cos \epsilon px \sin \epsilon py \, dp. \tag{45}$$

In [43]–[45], $N_1 = 2\mu D_s/n_0 kTa$ and $N_2 = (\eta + \kappa)D_s/n_0 kTa^2$, two of the dimensionless groups introduced in I. Note that the kernels $K_1$ and $K_2$ are symmetric.

We next calculate the drag force $-F\mathbf{z}$ on the disk, where

$$F = -\int_{S_1} [\tau_{zz}]_{h-}^{h+} \rho \, d\rho \, d\phi, \tag{46}$$

and the integral is taken over one side $S_1$ of the disk. Using [28] this can be written as

$$F = 8\pi\mu \int_0^1 \left(\frac{\partial\psi}{\partial z}\right)_{h+} \rho \, d\rho, \tag{47}$$

and from [16] and [35],

$$\left(\frac{\partial\psi}{\partial z}\right)_{h+} = -\frac{1}{\rho}\frac{d}{d\rho}\left[\rho \int_0^a g(t)\left\{\int_0^\infty J_1(s\rho)\cos st \, ds\right\} dt\right]. \tag{48}$$

Substituting [48] into [47], interchanging the orders of integration and performing the $\rho$-integration leads to

$$\begin{aligned} F &= -8\pi\mu a \int_0^a g(t)\left\{\int_0^\infty J_1(sa)\cos st \, ds\right\} dt \\ &= -8\pi\mu \int_0^a g(t)\, dt. \end{aligned} \tag{49}$$

In terms of the non-dimensionalisation of the previous paragraph, [49] reads

$$F = -8\pi\mu a U \int_0^1 G(x)\, dx, \tag{50}$$

where $G(x)$ is found by a numerical integration of [41] and [42]. We note that in the limit $h \to \infty$, i.e. $\epsilon \to 0$, [41] and [42] have the solutions

$$G(x) = -2/\pi, J(x) = 0,$$

and from [50]

$$F_\infty = 16\mu a U,$$

the known infinite-fluid result.

### 3.2 *Adsorption from the desorbtion to the bulk fluid*

This problem differs from that of 3.1 in that condition [6] is replaced by [7]; hence all that is necessary is to set $\alpha = 0$ and replace $\eta + \kappa$ by $\eta + \kappa + n_0 kT\beta^{-1}$ in the previous subsection. The resulting governing integral equations retain the forms [41] and [42] with the kernels $K_i(x, y)$, $i = 1, \ldots, 3$, now defined by

$$K_1(x, y) = \frac{2\epsilon}{\pi} \int_0^\infty \left\{1 + 2p + 2p^2 - \frac{2p^2}{1 + \epsilon N_3 p}\right\} e^{-2p} \cos \epsilon px \cos \epsilon py \, dp, \tag{51}$$

$$K_2(x, y) = \frac{2\epsilon}{\pi} \int_0^\infty \left\{1 - 2p + 2p^2 - \frac{2(1-p)^2}{1 + \epsilon N_3 p}\right\} e^{-2p} \sin \epsilon px \sin \epsilon py \, dp, \tag{52}$$

and

$$K_3(x, y) = \frac{4\epsilon}{\pi} \int_0^\infty \left\{p^2 + \frac{p(1-p)}{1 + \epsilon N_3 p}\right\} e^{-2p} \cos \epsilon px \sin \epsilon py \, dp. \tag{53}$$

In [51]–[53], $N_3 = (n_0 kT\beta^{-1} + \eta + \kappa)/2\mu a$, as defined in I. Again the drag is given by [50].

3.3 *Bulk diffusion into the film*

We consider first the boundary-value problem for the perturbation concentration $c' = n'/h_0$ of the soluble surfactant, which satisfies Laplace's equation [8]. In the case of an impervious sealed disk, there is zero flux of solute across the disk; thus the flux $\partial n'/\partial z$ must be continuous across the plane $z = h$, and zero on the disk. Further $c'$ must be continuous across $z = h$ for $\rho > a$ and tend to zero as $\rho^2 + z^2 \to \infty$, and we have the mixed conditions

$$\frac{\partial n'}{\partial z} = 0 \text{ on } z = h, 0 \leq \rho < a, \tag{54}$$

$$n' = 0 \text{ on } z = h, \rho > a, \tag{55}$$

with $n' \to 0$ as $\rho^2 + z^2 \to \infty$.

A suitable representation for $n'$ is

$$n' = \int_0^\infty \{P(s) e^{-s(z-h)} + Q(s) e^{s(z-h)}\} J_0(s\rho) \, ds, 0 \leq z < h, \tag{56}$$

$$n' = \int_0^\infty R(s) J_0(s\rho) e^{-s(z-h)} \, ds, z > h, \tag{57}$$

and continuity of flux across $z = h$ requires that $R = P - Q$. The mixed conditions [54] and [55] then lead to the dual integral equations

$$\int_0^\infty sQ(s) J_0(s\rho) \, ds = \int_0^\infty sP(s) J_0(s\rho) \, ds, 0 \leq \rho < a, \tag{58}$$

and

$$\int_0^\infty Q(s) J_0(s\rho) \, ds = 0, \rho > a. \tag{59}$$

To solve [58] and [59] we follow Sneddon (1966) and put

$$Q(s) = \int_0^a l(t) \sin st \, dt, l(0) = 0. \tag{60}$$

which satisfies [59] identically. The substitution of [60] into [58] and standard manipulations then give the relation

$$l(t) = \frac{2}{\pi} \int_0^{\infty} P(s) \sin st \, ds, \; 0 \le t \le a. \qquad [61]$$

Turning next to the conditions on the surface, [9], [25] and [56] give

$$n_0[hs\{(A + B) e^{-sh} + C e^{sh}\} - B e^{-sh} - sD e^{sh}] = \frac{D_0}{h_0}(-P e^{sh} + Q e^{-sh}), \qquad [62]$$

whilst [10] and [11] supply

$$\begin{aligned} 2\mu\{(A + B) e^{-sh} + C e^{sh}\} &= kT(P e^{sh} + Q e^{-sh}) \\ &- (\eta + \kappa)[hs\{(A + B) e^{-sh} + C e^{sh}\} - B e^{-sh} - sD e^{sh}]. \end{aligned} \qquad [63]$$

The conditions on $z = 0$ are completed by [5] resulting in [30], and [30], [62] and [63] are now solved to express $C$, $D$ and $P$ in terms of $A$, $B$ and $Q$. With $C$, $D$ and $P$ written in terms of $g(t)$, $j(t)$ and $l(t)$, [37], [38] and [61] then provide a set of three simultaneous Fredholm equations for $g(t)$, $j(t)$ and $l(t)$. Using the non-dimensionalisation of the previous sections and defining $L(x) = D_0 l(t)/U h_0 n_0$, these equations take the forms

$$G(x) - \int_0^1 K_1(x, y) G(y) \, dy = -\frac{2}{\pi} - \int_0^1 K_4(x, y) J(y) \, dy - N_4 \int_0^1 K_5(x, y) L(y) \, dy, \qquad [64]$$

$$J(x) - \int_0^1 K_2(x, y) J(y) \, dy = -\int_0^1 K_4(y, x) G(y) \, dy - N_4 \int_0^1 K_6(x, y) L(y) \, dy, \qquad [65]$$

and

$$L(x) - \int_0^1 K_3(x, y) L(y) \, dy = -\int_0^1 K_5(y, x) G(y) \, dy - \int_0^1 K_6(x, y) J(y) \, dy, \quad 0 \le x \le 1. \qquad [66]$$

The kernels $K_i(x, y)$, $i = 1, \ldots, 6$, are defined by

$$K_1(x, y) = \frac{2\epsilon}{\pi} \int_0^{\infty} \left\{ 1 + 2p + 2p^2 - \frac{2p^2}{1 + N_4 + \epsilon N_5 p} \right\} e^{-2p} \cos \epsilon px \cos \epsilon py \, dp, \qquad [67]$$

$$K_2(x, y) = \frac{2\epsilon}{\pi} \int_0^{\infty} \left\{ 1 - 2p + 2p^2 - \frac{2(1 - p)^2}{1 + N_4 + \epsilon N_5 p} \right\} e^{-2p} \sin \epsilon px \sin \epsilon py \, dp, \qquad [68]$$

$$K_3(x, y) = \frac{2\epsilon}{\pi} \int_0^{\infty} \left\{ 1 - \frac{2N_4}{1 + N_4 + \epsilon N_5 p} \right\} e^{-2p} \sin \epsilon px \sin \epsilon py \, dp, \qquad [69]$$

$$K_4(x, y) = \frac{4\epsilon}{\pi} \int_0^{\infty} \frac{p\{1 + p(N_4 + \epsilon p)\}}{1 + N_4 + \epsilon N_5 p} e^{-2p} \cos \epsilon px \sin \epsilon py \, dp, \qquad [70]$$

$$K_5(x, y) = \frac{4\epsilon}{\pi}\int_0^\infty \frac{p}{1 + N_4 + \epsilon N_5 p}\, e^{-2p} \cos \epsilon p x \sin \epsilon p y \, dp, \qquad [71]$$

and

$$K_6(x, y) = \frac{4\epsilon}{\pi}\int_0^\infty \frac{1-p}{1 + N_4 + \epsilon N_5 p}\, e^{-2p} \sin \epsilon p x \sin \epsilon p y \, dp. \qquad [72]$$

In [67]–[72] $N_4$ and $N_5$ are the final pair of dimensionless groups introduced in I, with $N_4 = h_0 n_0 kT/2\mu D_0$ and $N_5 = (\eta + \kappa)/2\mu a$, and the drag on the disk is again furnished by [50].

Similar formulations can be given for other linear boundary conditions applied to the solute concentration at the disk; for example the solute concentration may be maintained at the equilibrium concentration $c_0$. This leads to a slight modification of the dual integral equations [58] and [59], but it is straightforward to derive the governing set of three Fredholm integral equations of the second kind.

## 4. ITERATIVE AND NUMERICAL RESULTS FOR THE DISK

None of the governing integral equations formulated in the previous section are capable of exact analytic solution. Thus two possible approaches are asymptotic solution, valid when one or more of the parameters in the problem are small, and a full numerical treatment. In the following paragraphs both these options are used.

### 4.1 *Surface diffusion model*

The governing integral equations are [41] and [42] with the various kernel functions given in [43]–[45]. As a first example of the asymptotic approach, suppose that $\epsilon = a/h \ll 1$, and both of the parameters $N_1$, $N_2$ are of order unity. The kernels [43]–[45] are easily expanded in power series in $\epsilon$, and the first few terms in the Neumann iterative solution of [41] and [42] are found as

$$G(x) = \frac{2}{\pi}\left\{1 + \frac{3}{\pi}\epsilon + \frac{1}{\pi}\left(\frac{9}{\pi} - \frac{3N_1}{2}\right)\epsilon^2 + \frac{1}{\pi}\left(\frac{27}{\pi^2} - \frac{5}{6} - \frac{9N_1}{\pi} + 3N_1^2 - \frac{5x^2}{2}\right)\epsilon^3 + O(\epsilon^4)\right\}, \qquad [73]$$

and

$$J(x) = \frac{3}{\pi^2} x\epsilon^2\left\{1 - \frac{1}{\pi}\left(\frac{3}{\pi} - N_1\right)\epsilon\right\} + O(\epsilon^4). \qquad [74]$$

The drag ratio $F/F_\infty$ then follows from [50] and [73] as

$$\frac{F}{F_\infty} = 1 + \frac{3}{\pi}\epsilon + \frac{1}{\pi}\left(\frac{9}{\pi} - \frac{3N_1}{2}\right)\epsilon^2 + \frac{1}{\pi}\left(\frac{27}{\pi^2} - \frac{5}{3} - \frac{9N_1}{\pi} + 3N_1^2\right)\epsilon^3 + O(\epsilon^4), \qquad [75]$$

and is independent of $N_2$ to this order of approximation.

The first three terms in [75] confirm the asymptotic result predicted by [53] in I. When $\epsilon = 0.1$, with $N_1 = N_2 = 1$, [75] gives $F/F_\infty = 1.1045$, whereas the numerically computed ratio is 1.1044. Indeed [75] may be confidently used for $\epsilon$ as large as 0.4, with an error of the order of only a few per cent. For larger values of $\epsilon$ the accuracy of [75] falls off rapidly. A considerable number of further iterative results are easily obtained for $\epsilon \ll 1$ with $N_1$ and $N_2$ of various orders in $\epsilon$, and we quote two further examples. Inspection of the integral equations shows that [75] is valid when $N_1 = O(1)$ and $N_2 = O(\epsilon)$. For $N_1 = O(1)$, $N_2 = O(\epsilon^{-1})$ and $\Lambda_1 = \epsilon N_2 = O(1)$, calculation shows that the formula [75] is changed by the addition of a term $15\Lambda_1 N_1/2$ in the bracketed coefficient of $\epsilon^3$.

In order to carry out the numerical integration of [41] and [42] for varying values of $a/h$, $N_1$ and $N_2$, a NAG library routine based on El-gendi's (1969) method was used. Details of this method can be found in Shail & Gooden (1981) and Chakrabarti, Gooden & Shail (1982), and it will not be described further here. To exhibit the effect of the surfactant film on the drag ratio $F/F_\infty$ we present the results of numerical computations for $\epsilon = 1.0$ and various values of $N_1$ and $N_2$ In table 1 sample values of $F/F_\infty$ are given, and figure 1 shows on a log-linear plot the variation of $F/F_\infty$ with $N_1$ for specimen values of $N_2$. Inspection of the table reveals that for fixed $N_1$ and increasing $N_2$ the drag ratio $F/F_\infty$ increases monotonically, whereas for fixed $N_2$ and increasing $N_1$ the ratio decreases monotonically. This is to be expected since as $N_2 \to \infty$ the surfactant responds increasingly as a rigid bounding plane, whereas when $N_1 \to \infty$ the response is that of a free uncontaminated surface. In the limit of a rigid bounding surface, $F/F_\infty = 3.1213$, and for a free surface the corresponding ratio is 2.0536.

For smaller values of $a/h$ the effects of the surfactant diminish and for $\epsilon = 0.1$ there is only a variation of the order of 4% over the ranges $0 \le N_1$, $N_2 < \infty$; limiting values are $F/F_\infty = 1.1049$ for a rigid bounding plane, and $F/F_\infty = 1.0678$ for a free surface.

Table 1. Values of $F/F_\infty$ for $a/h = 1.0$ and various values of $N_1$, $N_2$

| $N_2$ \ $N_1$ | 0.025 | 0.05 | 0.1 | 0.25 | 0.5 | 1.0 |
|---|---|---|---|---|---|---|
| 0.00 | 3.0795 | 3.0412 | 2.9738 | 2.8222 | 2.6631 | 2.4916 |
| 0.025 | 3.0813 | 3.0446 | 2.9794 | 2.8308 | 2.6725 | 2.4997 |
| 0.05 | 3.0830 | 3.0476 | 2.9844 | 2.8386 | 2.6813 | 2.5076 |
| 0.1 | 3.0857 | 3.0526 | 2.9929 | 2.8527 | 2.6975 | 2.5225 |
| 0.25 | 3.0915 | 3.0633 | 3.0116 | 2.8853 | 2.7378 | 2.5618 |
| 0.5 | 3.0972 | 3.0742 | 3.0313 | 2.9221 | 2.7868 | 2.6144 |
| 1.0 | 3.1034 | 3.0861 | 3.0531 | 2.9659 | 2.8504 | 2.6901 |
| 1.5 | 3.1068 | 3.0927 | 3.0656 | 2.9921 | 2.8911 | 2.7432 |
| 2.0 | 3.1090 | 3.0970 | 3.0738 | 3.0101 | 2.9200 | 2.7832 |
| 5.0 | 3.1147 | 3.1081 | 3.0952 | 3.0584 | 3.0027 | 2.9084 |
| 10.0 | 3.1174 | 3.1136 | 3.1059 | 3.0837 | 3.0487 | 2.9859 |
| 50.0 | 3.1204 | 3.1194 | 3.1175 | 3.1119 | 3.1026 | 3.0846 |
| 100.0 | 3.1208 | 3.1203 | 3.1194 | 3.1164 | 3.1115 | 3.1018 |

| $N_2$ \ $N_1$ | 1.5 | 2.0 | 5.0 | 10.0 | 50.0 | 100.0 |
|---|---|---|---|---|---|---|
| 0.00 | 2.3985 | 2.3659 | 2.1970 | 2.1331 | 2.0713 | 2.0626 |
| 0.025 | 2.4053 | 2.3451 | 2.2000 | 2.1347 | 2.0716 | 2.0628 |
| 0.05 | 2.4120 | 2.3508 | 2.2030 | 2.1363 | 2.0720 | 2.0630 |
| 0.1 | 2.4248 | 2.3618* | 2.2089 | 2.1396 | 2.0727 | 2.0633 |
| 0.25 | 2.4597 | 2.3925 | 2.2259 | 2.1492 | 2.0748 | 2.0644 |
| 0.5 | 2.5087 | 2.4371 | 2.2524 | 2.1647 | 2.0784 | 2.0662 |
| 1.0 | 2.5840 | 2.5085 | 2.2999 | 2.1937 | 2.0853 | 2.0698 |
| 1.5 | 2.6400 | 2.5639 | 2.3413 | 2.2206 | 2.0921 | 2.0733 |
| 2.0 | 2.6840 | 2.6087 | 2.3779 | 2.2456 | 2.0988 | 2.0768 |
| 5.0 | 2.8317 | 2.7679 | 2.5343 | 2.3663 | 2.1368 | 2.0971 |
| 10.0 | 2.9310 | 2.8825 | 2.6789 | 2.5006 | 2.1929 | 2.1288 |
| 50.0 | 3.0674 | 3.0508 | 2.9630 | 2.8498 | 2.4674 | 2.3168 |
| 100.0 | 3.0924 | 3.0832 | 3.0318 | 2.9584 | 2.6346 | 2.4626 |

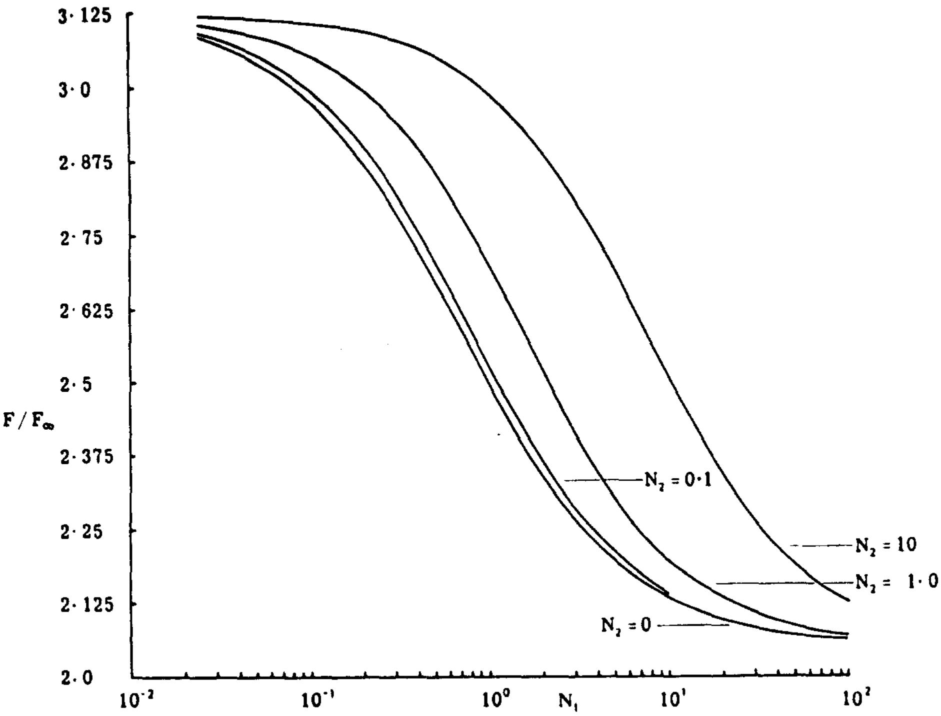


Figure 1. Log-linear plots of $F/F_\infty$ against $N_1$ for $N_2 = 0, 0.1, 1.0$, and 10 in the surface diffusion case for the disk with $a/h = 1.0$.

4.2 *Adsorption/desorption model*

The integral equations to be solved are [41] and [42] with the kernel functions given by [51]–[53]. Consider first the iterative solution when $\epsilon = a/h \ll 1$. In the case when $N_3 = O(1)$, it is found that

$$G(x) = \frac{2}{\pi}\left\{1 + \frac{2}{\pi}\epsilon + \frac{1}{\pi}\left(\frac{4}{\pi} + \frac{3N_3}{2}\right)\epsilon^2 + \frac{1}{\pi}\left(\frac{8}{\pi^2} - \frac{1}{3} + \frac{6N_3}{\pi} - 3N_3^2 - x^2\right)\epsilon^3\right\} + O(\epsilon^4), \quad [76]$$

$$J(x) = \frac{2}{\pi^2}x\epsilon^2\left\{1 - \left(\frac{2}{\pi} + 3N_3\right)\epsilon\right\} + O(\epsilon^4), \quad [77]$$

and

$$\frac{F}{F_\infty} = 1 + \frac{2}{\pi}\epsilon + \frac{1}{\pi}\left(\frac{4}{\pi} + \frac{3N_3}{2}\right)\epsilon^2 + \frac{1}{\pi}\left(\frac{8}{\pi^2} - \frac{2}{3} + \frac{6N_3}{\pi} - 3N_3^2\right)\epsilon^3 + O(\epsilon^4), \quad [78]$$

the first three terms in [78] agreeing with the result derived from [59] in I. For $N_3 = 1$ and $\epsilon = 0.1$, [78] predicts that $F/F_\infty = 1.0722$, the numerically computed value being 1.0723.

As a second illustrative iterative solution, suppose that $N_3 = O(\epsilon^{-1})$ and set $\Lambda_2 = \epsilon N_3 = O(1)$. We then find that

$$G(x) = \frac{2}{\pi}\left[1 + \frac{2\alpha}{\pi}\epsilon + \frac{4\alpha^2}{\pi^2}\epsilon^2 + \frac{1}{\pi}\left\{\frac{8\alpha^3}{\pi^2} - \left(x^2 + \frac{1}{3}\right)\beta\right\}\epsilon^3 + O(\epsilon^4)\right], \quad [79]$$

$$J(x) = \frac{4}{\pi}x\gamma\epsilon^2\left(1 - \frac{2\alpha}{\pi}\epsilon\right) + O(\epsilon^4), \quad [80]$$

and

$$\frac{F}{F_\infty} = 1 + \frac{2\alpha}{\pi}\epsilon + \frac{4\alpha^2}{\pi^2}\epsilon^2 + \frac{2}{\pi}\left(\frac{4\alpha^2}{\pi^2} - \frac{\beta}{3}\right)\epsilon^3 + O(\epsilon^4), \qquad [81]$$

where $\alpha$, $\beta$, $\gamma$ are expressed in terms of the exponential integral $E_1(y)$ (Abramowitz & Stegun 1964) as

$$\alpha = \frac{3}{2} - \frac{1}{2\Lambda_2} + \frac{1}{\Lambda_2^2} - \frac{2}{\Lambda_2^3}e^{2/\Lambda_2}E_1(2/\Lambda_2),$$

$$\beta = \frac{5}{2} - \frac{3}{4\Lambda_2} + \frac{1}{2\Lambda_2^2} - \frac{1}{2\Lambda_2^3} + \frac{1}{\Lambda_2^4} - \frac{2}{\Lambda_2^5}e^{2/\Lambda_2}E_1(2/\Lambda_2),$$

and

$$\gamma = \frac{3}{4} - \frac{1}{2\Lambda_2^2} - \frac{1}{\Lambda_2^3} + \frac{2(\Lambda_2+1)}{\Lambda_2^4}e^{2/\Lambda_2}E_1(2/\Lambda_2).$$

For $\epsilon = 0.1$ and $\Lambda_2 = 1$, [81] supplies the result $F/F_\infty = 1.0880$ which agrees to 4 decimal places with the numerically computed result.

We next present some results of a numerical integration of [41] and [42] for $a/h = 1.0$, $a/h = 0.1$, and various values of $N_3$. The values obtained for $F/F_\infty$ are given in table 2, and figure 2 shows log-linear plots of $F/\mu aU$ against $N_3$ for $a/h = 1.0$ and 0.1. The values of $F/\mu aU$ for fixed $\epsilon$ again increase monotonically with $N_3$, $N_3 = 0$ constituting a free uncontaminated surface and the limit $N_3 \to \infty$ giving a rigid bounding plane.

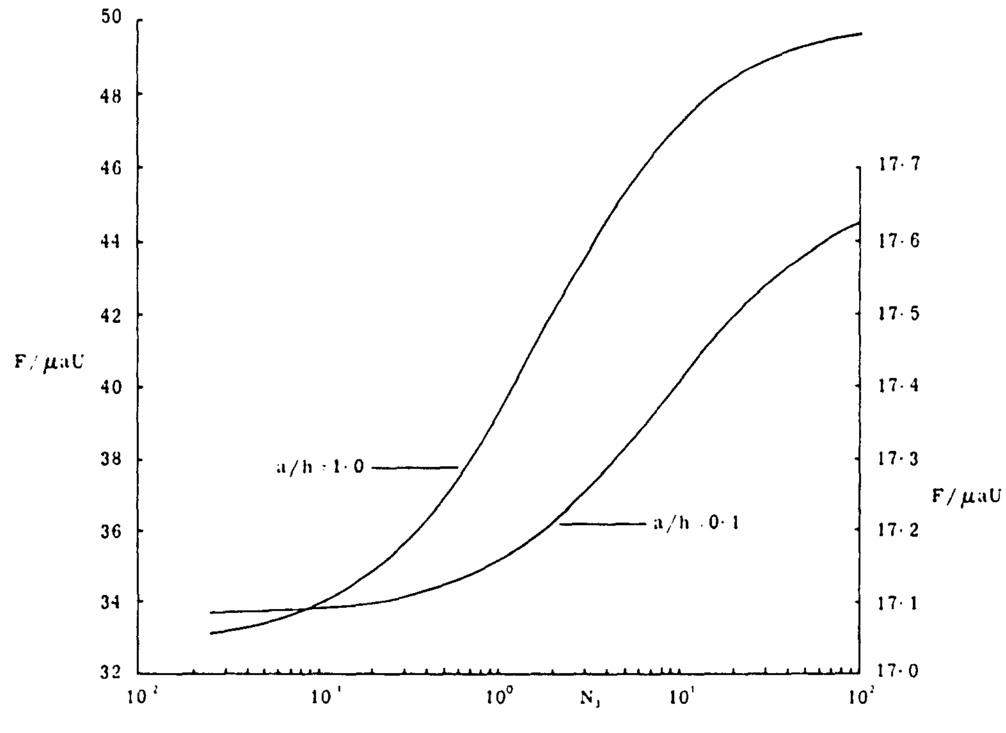


Figure 2. Log-linear plots of $F/\mu aU$ against $N_3$ for $a/h = 1.0$ and 0.1 in the adsorbtion/desorbtion case for the disk. The left-hand vertical scale pertains to $a/h = 1.0$, the right-hand scale to $a/h = 0.1$.

Table 2. Values of $F/F_\infty$ for $a/h = 1.0$, $0.1$ and various values of $N_3$

| $N_3$ | a/h = 1.0 | a/h = 0.1 | $N_3$ | a/h = 1.0 | a/h = 0.1 |
|---|---|---|---|---|---|
| 0.0 | 2.0536 | 1.0678 | 1.75 | 2.5948 | 1.0749 |
| 0.025 | 2.0714 | 1.0679 | 2.0 | 2.6280 | 1.0756 |
| 0.05 | 2.0885 | 1.0680 | 3.0 | 2.7261 | 1.0783 |
| 0.075 | 2.1048 | 1.0681 | 4.0 | 2.7907 | 1.0804 |
| 0.1 | 2.1205 | 1.0683 | 5.0 | 2.8367 | 1.0822 |
| 0.25 | 2.2033 | 1.0690 | 6.0 | 2.8713 | 1.0837 |
| 0.5 | 2.3104 | 1.0702 | 7.0 | 2.9198 | 1.0861 |
| 0.75 | 2.3923 | 1.0713 | 10.0 | 2.9524 | 1.0880 |
| 1.0 | 2.4577 | 1.0723 | 50.0 | 3.0810 | 1.0990 |
| 1.25 | 2.5113 | 1.0732 | 100.0 | 3.1006 | 1.1016 |
| 1.5 | 2.5564 | 1.0741 | ∞ | 3.1213 | 1.1049 |

4.3 *Bulk diffusion model*

In this case the three simultaneous integral equations [64]–[66], with kernels [67]–[72] are to be solved. Numerical solution by El-gendi's methd is entirely feasible, but has not been carried out to date owing to limited resources. Thus we content ourselves with some observations on the iterative solution when $\epsilon \ll 1$ and $N_4$, $N_5$ are of order unity. A somewhat lengthy calculation then produces the Neumann series

$$G(x) = -\frac{2}{\pi} - \frac{2(3N_4+2)}{\pi^2(N_4+1)}\epsilon - \frac{2}{\pi^2}\left\{\frac{3N_5}{2(N_4+1)^2} + \frac{(3N_4+2)^2}{\pi(N_4+1)^2}\right\}\epsilon^2 + \frac{2}{\pi^2}\left\{-\frac{3N_5(3N_4+2)}{\pi(N_4+1)^3}\right.$$

$$\left. - \frac{(3N_4+2)^3}{\pi^2(N_4+1)^3} + \frac{3N_5^2}{(N_4+1)^2} + \frac{5N_4+2}{6(N_4+1)} + \frac{5N_4+2}{2(N_4+1)}x^2\right\} + O(\epsilon^4), \qquad [82]$$

$$J(x) = \frac{(3N_4+2)}{\pi^2(N_4+1)}x\epsilon^2 + \frac{1}{\pi^2(N_4+1)^2}\left\{\frac{(3N_4+2)^2}{\pi} + 3N_5(2N_4+1) - 6(N_4+1)\right\}\epsilon^3 + O(\epsilon^4), \qquad [83]$$

and

$$L(x) = \frac{2x}{\pi^2(N_4+1)}\epsilon^2 + \frac{x}{\pi^2(N_4+1)^2}\left\{\frac{2(3N_4+2)}{\pi} - 3N_5\right\}\epsilon^3 + O(\epsilon^4). \qquad [84]$$

Further, the drag ratio is found as

$$\frac{F}{F_t} = 1 + \frac{(3N_4+2)}{\pi(N_4+1)}\epsilon + \frac{1}{\pi(N_4+1)^2}\left\{\frac{3N_5}{2} + \frac{(3N_4+2)^2}{\pi}\right\}\epsilon^2 + \frac{1}{\pi(N_4+1)^3}$$

$$\times\left\{\frac{3N_5(3N_4+2)}{\pi} + \frac{(3N_4+2)^3}{\pi^2} - \frac{5}{3}(N_4+2)(N_4+1)^2\right\}\epsilon^3 + O(\epsilon^4). \qquad [85]$$

It is of interest to note that to $O(\epsilon^2)$, [85] agrees with the result of [63] in I, and we conjecture that [63] of I remains valid for an arbitrary body, despite the caveat in I.

However, if the Neumann condition on $n'$ at the disk surface is replaced by the Dirichlet condition $n' = 0$, then after carrying out the iterative solution the drag ratio is obtained as

$$\frac{F}{F_\infty} = 1 + \frac{(3N_4 + 2)}{\pi(N_4 + 1)}\epsilon + \frac{1}{\pi(N_4 + 1)^2}\left\{\frac{3N_5}{2} + \frac{(3N_4 + 2)^2}{\pi} - \frac{N_4}{\pi}\right\} + O(\epsilon^3). \quad [86]$$

The final brace in [86] differs from that in [63] of I by the term $-N_4/\pi$, and therefore in this instance the effect of the translating disk on solute concentration cannot be ignored to $O(\epsilon^2)$.

## 5. THE TRANSLATING SPHERE PROBLEM

Suppose now that the translating body is a rigid sphere of radius $a$, whose centre is at an instantaneous depth $h$ below the surface $z = 0$. We assume that the disk is completely submerged so that $a/h < 1$. The geometrical configuration is best described in terms of bispherical coordinates $(\xi, \eta, \phi)$, related to cylindrical coordinates $(\rho, \phi, z)$ by

$$\rho = \frac{c \sin \eta}{\cosh \xi - \cos \eta}, \quad z = \frac{c \sinh \xi}{\cosh \xi - \cos \eta}, \quad [87]$$

where $c > 0$ is a scale parameter, $-\infty < \xi < \infty$, and $0 \le \eta \le \pi$. The curves $\xi$ = constant are a family of coaxial spheres having $z = 0$ (i.e. $\xi = 0$) as radical plane. The translating sphere is given by $\xi = \beta(>0)$, and its distance $h$ from the plane and radius $a$ are

$$h = c \coth \beta, \qquad a = c \operatorname{cosech} \beta. \quad [88]$$

The fluid motion is conveniently expressed in terms of a stream function $\psi$, the velocity components being

$$u = \frac{1}{\rho}\frac{\partial \psi}{\partial z}, \quad w = -\frac{1}{\rho}\frac{\partial \psi}{\partial \rho}, \quad [89]$$

and in bispherical coordinates $\psi$ satisfies the equation

$$M^4 \psi = 0, \quad [90]$$

where, with $s = \cos \eta$,

$$M^2 = \frac{(\cosh \xi - s)}{c^2}\left\{\frac{\partial}{\partial \xi}\left((\cosh \xi - s)\frac{\partial}{\partial \xi}\right) + (1 - s^2)\frac{\partial}{\partial s}\left((\cosh \xi - s)\frac{\partial}{\partial s}\right)\right\}. \quad [91]$$

As originally shown by Stimson & Jeffery (1926) (see also Brenner 1961), the appropriate general solution of [90] has the form

$$\psi = (\cosh \xi - s)^{-\frac{3}{2}} \sum_{n=1}^{\infty} U_n(\xi) C_{n+1}^{-\frac{1}{2}}(s), \quad [92]$$

where

$$U_n(\xi) = a_n \cosh (n - \tfrac{1}{2})\xi + b_n \sinh (n - \tfrac{1}{2})\xi + c_n \cosh (n + \tfrac{3}{2})\xi + d_n \sinh (n + \tfrac{3}{2})\xi, \quad [93]$$

and $C_{n+1}^{-\frac{1}{2}}(s)$ is the Gegenbauer polynomial of order $n+1$ and degree $-\frac{1}{2}$, defined in terms of Legendre polynomials by

$$C_{n+1}^{-\frac{1}{2}}(s) = (P_{n-1}(s) - P_{n+1}(s))/2n+1. \qquad [94]$$

The $C_{n+1}^{-\frac{1}{2}}(s)$, $n = 1, 2, \ldots$, form a complete orthogonal set on $[-1, 1]$, with the orthogonality relation

$$\int_{-1}^{1} \frac{C_{n+1}^{-\frac{1}{2}}(x)\, C_{m+1}^{-\frac{1}{2}}(x)}{1-x^2}\, \mathrm{d}x = \frac{2}{n(n+1)(2n+1)}\delta_{nm}. \qquad [95]$$

It also follows from the work of Stimson and Jeffery that in terms of the coefficients $a_n, \ldots, d_n$, the drag force on the sphere is $-F\mathbf{z}$, where

$$F = -\frac{2\sqrt{2}\pi\mu}{c}\sum_{n=1}^{\infty}(a_n + b_n + c_n + d_n). \qquad [96]$$

We next determine the coefficients in [96] from the surfactant and no-slip boundary conditions. Condition [5] requires that $\partial\psi/\partial\rho = 0$ on $z = 0$, which can be satisfied by setting $\psi = 0$ on $\xi = 0$. From [92], [93] and the completeness of the $C_{n+1}^{-\frac{1}{2}}(s)$ this implies that

$$a_n + c_n = 0, \quad n = 1, 2, \ldots \qquad [97]$$

On the sphere $\xi = \beta$ Stimson and Jeffery point out that the no-slip conditions are equivalent to

$$\psi + \tfrac{1}{2}\rho^2 U = 0 \quad \text{and} \quad \frac{\partial}{\partial\xi}(\psi + \tfrac{1}{2}\rho^2 U) = 0, \qquad [98]$$

which, using [92] and [97], leads to

$$\begin{aligned}\Delta_n b_n = &-c_n\{2n+3-(2n+1)\cosh 2\beta - 2\cosh(2n+1)\beta\}\\ &-\delta_n(2n+3)\{(2n+1)\,\mathrm{e}^{2\beta} + 2\,\mathrm{e}^{-(2n+1)\beta} - 2n+1\},\end{aligned} \qquad [99]$$

and

$$\begin{aligned}\Delta_n d_n = &-c_n\{2n-1-(2n+1)\cosh 2\beta + 2\cosh(2n+1)\beta\}\\ &-\delta_n(2n-1)\{(2n+1)\,\mathrm{e}^{-2\beta} - 2\,\mathrm{e}^{-(2n+1)\beta} - 2n-3\},\end{aligned} \qquad [100]$$

where

$$\Delta_n = 2\sinh(2n+1)\beta - (2n+1)\sinh 2\beta \qquad [101]$$

and

$$\delta_n = c^2 U n(n+1)/(2n-1)(2n+3)\sqrt{2}. \qquad [102]$$

From [96], [97], and [99]–[102] the drag ratio $F/F_\infty$, where $F_\infty = 6\pi\mu a U$, is now found as

$$\frac{F}{F_\infty} = \frac{4}{3}\sinh\beta \sum_{n=1}^{\infty} \frac{(2n+1)}{\Delta_n}\bar{c}_n + \frac{4}{3}\sinh\beta \sum_{n=1}^{\infty} \frac{n(n+1)}{(2n-1)(2n+3)}$$

$$\times \left\{\frac{4\cosh^2(n+\frac{1}{2})\beta + (2n+1)^2\sinh^2\beta}{\Delta_n} - 1\right\}, \qquad [103]$$

where $\bar{c}_n = -c_n\sqrt{2}/Ua^2$. Equations for the determination of $\bar{c}_n$ now follow from the remaining film boundary conditions, which we consider in the surface diffusion and adsorption/desorption situations.

### 5.1 *Surface diffusion model*

The appropriate film boundary condition is [6], and substituting [89] and [92] into [6] gives the nondimensionalised equation

$$-2N_1(1-s)\sum_{n=1}^{\infty}(2n+1)\bar{c}_n C_{n+1}^{-\frac{1}{2}}(s) = -4\sinh^2\beta \sum_{n=1}^{\infty}(\tau_n\bar{c}_n + \kappa_n)C_{n+1}^{-\frac{1}{2}}(s) + (1-s)N_2$$

$$\times \sum_{n=1}^{\infty}(\tau_n\bar{c}_n + \kappa_n)\{(n-1)(2n-1)C_n^{-\frac{1}{2}}(s) - (2n+1)^2 C_{n+1}^{-\frac{1}{2}}(s)$$

$$+ (n+2)(2n+3)C_{n+2}^{-\frac{1}{2}}(s)\}, \quad -1 \le s \le 1, \qquad [104]$$

where

$$\kappa_n = \sinh^3\beta\, n(n+1)(2n+1)/\Delta_n, \qquad [105]$$

and

$$\tau_n = \{(2n+1)^2\sinh^2\beta - 4\sinh^2(n+\tfrac{1}{2})\beta\}/\Delta_n \sinh\beta. \qquad [106]$$

Defining $\bar{c}_n \equiv 0$ for $n \le 0$, the orthogonality relation [95] and the recurrence relation

$$(2n+1)sC_{n+1}^{-\frac{1}{2}}(s) = (n-1)C_n^{-\frac{1}{2}}(s) + (n+2)C_{n+2}^{-\frac{1}{2}}(s)$$

lead to the following infinite banded set of linear equations for the coefficients $\bar{c}_n$, $n = 1, 2, \ldots$:

$$N_2(n+2)(n+3)\{(\tau_{n+4}\bar{c}_{n+4} + \kappa_{n+4}) - 4(\tau_{n+3}\bar{c}_{n+3} + \kappa_{n+3})$$

$$+ 6(\tau_{n+2}\bar{c}_{n+2} + \kappa_{n+2}) - 4(\tau_{n+1}\bar{c}_{n+1} + \kappa_{n+1}) + (\tau_n\bar{c}_n + \kappa_n)\}$$

$$+ 2N_1\{(n+2)\bar{c}_{n+3} - (2n+5)\bar{c}_{n+2} + (n+3)\bar{c}_{n+1}\}$$

$$+ 4\sinh^2\beta(\tau_{n+2}\bar{c}_{n+2} + \kappa_{n+2}) = 0, \quad n \ge 0. \qquad [107]$$

A further relation between the $\bar{c}_n$-coefficients, useful in checking numerical work, can be derived from [104]. Dividing [104] through by $1-s$ and using the results

$$C_{n+1}^{-\frac{1}{2}}(1) = 0, \quad \lim_{s\to 1-} C_{n+1}^{-\frac{1}{2}}(s)/(1-s) = 1, \quad n \ge 1,$$

it follows that on letting $s \to 1-$,

$$\sum_{n=1}^{\infty} (\tau_n \bar{c}_n + \kappa_n) = 0. \quad [108]$$

Since

$$\lim_{n\to\infty} \tau_n = -\operatorname{cosech}\beta \quad \text{and} \quad \lim_{n\to\infty} \kappa_n = 0,$$

it is apparent from [108] that $\lim_{n\to\infty} \bar{c}_n = 0$.

### 5.2 *Adsorption/desorption model*

The relevant film boundary condition is now [7]; as with the disk problem a set of linear equations can be derived from [107] by means of the limiting process $D_s \to 0$, with $\eta + \kappa$ replaced by $\eta + \kappa + n_0 kT\beta^{-1}$. However a simpler set of linear equations with a matrix of smaller bandwidth is obtained by direct application of [7]. It is found that

$$-2\sum_{n=1}^{\infty}(2n+1)\bar{c}_n C_{n+1}^{-\frac{1}{2}}(s) = N_3 \sum_{n=1}^{\infty}(\tau_n\bar{c}_n+\kappa_n)\{(n-1)(2n-1)C_n^{-\frac{1}{2}}(s)-(2n+1)^2C_{n+1}^{-\frac{1}{2}}(s)$$
$$+(n+2)(2n+3)C_{n+2}^{-\frac{1}{2}}(s)\}, \quad -1 \le s \le 1. \quad [109]$$

Orthogonality of the $C_{n+1}^{-\frac{1}{2}}(s)$ now yields the linear set

$$N_3\{(n+1)(\tau_{n+2}\bar{c}_{n+2}+\kappa_{n+2})-(2n+3)(\tau_{n+1}\bar{c}_{n+1}+\kappa_{n+1})$$
$$+(n+2)(\tau_n\bar{c}_n+\kappa_n)\}+2\bar{c}_{n+1}=0, \quad n \ge 0, \quad [110]$$

again defining $\bar{c}_0 \equiv 0$. Summation of equations [110] gives the useful relation

$$\sum_{n=1}^{\infty} \bar{c}_n = 0, \quad [111]$$

with $\lim_{n\to\infty} \bar{c}_n = 0$.

## 6. NUMERICAL RESULTS FOR THE SPHERE

For general values of the parameters $N_1$, $N_2$, and $N_3$ it is not possible to solve in closed form the recurrence relations for the $\bar{c}_n$ furnished by [107] and [110]. Thus the method of truncation to a finite set of simultaneous linear equations is employed. In practice retention of the first 60 equations was found to be entirely satisfactory from the convergence point of view; this amounts to setting $\bar{c}_n = 0$ for $n \ge 61$, and the closeness to zero of the finite sums obtained in [108] and [111] by replacing the upper limit by 60 provides a convincing check on the convergence and accuracy of the numerical process. In all cases the truncated set of banded simultaneous linear equations was solved using a library routine, followed by evaluation of the summations in the drag ratio formula [103].

### 6.1 *Surface diffusion model*

In order to vary the mode of numerical coverage of surfactant effects we present the results for the surface diffusion model in a different format from that in section 4.1 for the

Table 3. Values of $F/F_\infty$ for varying $h/a$ and $(N_1, N_2) = (1,1)$, (10,10) and (0.1,0.1)

| h/a | $N_1=N_2=1$ | $N_1=N_2=10$ | $N_1=N_2=0.1$ | clean Surface |
|---|---|---|---|---|
| 1.5 | 2.7277 | 2.5122 | 3.0582 | 2.0387 |
| 2.0 | 1.9389 | 1.8087 | 2.0813 | 1.5967 |
| 2.5 | 1.6515 | 1.5560 | 1.7350 | 1.4242 |
| 3.0 | 1.5006 | 1.4246 | 1.5567 | 1.3302 |
| 3.5 | 1.4072 | 1.3440 | 1.4478 | 1.2705 |
| 4.0 | 1.3435 | 1.2895 | 1.3744 | 1.2292 |
| 4.5 | 1.2972 | 1.2502 | 1.3215 | 1.1988 |
| 5.0 | 1.2620 | 1.2205 | 1.2817 | 1.1756 |
| 6.0 | 1.2120 | 1.1786 | 1.2257 | 1.1423 |
| 7.0 | 1.1781 | 1.1505 | 1.1882 | 1.1197 |
| 8.0 | 1.1536 | 1.1303 | 1.1614 | 1.1032 |
| 9.0 | 1.1350 | 1.1150 | 1.1412 | 1.0907 |
| 10.0 | 1.1205 | 1.1031 | 1.1255 | 1.0810 |

disk. In table 3 values of $F/F_\infty$ are displayed for a range of values of $\epsilon^{-1} = h/a$, and the pairs of values (1,1), (10,10), (0.1,0.1) for $(N_1, N_2)$. The final column gives $F/F_\infty$ for a free uncontaminated surface, obtained by letting $N_1 \to \infty$ for fixed $N_2$. Equations [107] then have the trivial solution $\bar{c}_n = \text{constant} = 0$, since $\bar{c}_n \to 0$ as $n \to \infty$. The ratio $F/F_\infty$ is given by the second summation in [103], a result in agreement with that of Brenner (1962). It is also worth noting that for a rigid bounding plane, letting $N_2 \to \infty$ with $N_1$ fixed gives

$$\tau_n \bar{c}_n + \kappa_n = \text{constant} = 0,$$

since $\kappa_n$, $\bar{c}_n \to 0$ as $n \to \infty$. The resulting expression for $F/F_\infty$ is in accord with that obtained by Brenner (1962) (the earlier result of Stimson & Jeffery (1926) contains misprints).

The variation of $F/F_\infty$ is further exhibited in figure 3 for the values (5, 0.5), (1,1), (0.5,5) of $(N_1, N_2)$, and for the clean free surface. The drag ratio tends to infinity as $h/a \to 1$, i.e. as the sphere approaches the surface. However it must be remembered that as $h/a \to 1$, the assumption that the surface remains plane will become invalid (further discussion of this assumption can be found in Bart 1968).

### 6.2 *Adsorbtion/desorbtion model*

On applying the truncation method to [110], numerical results are obtained for the adsorbtion/desorbtion film process. Again, [111] was used as a check on the convergence of the process. In table 4 values of the drag ratio $F/F_\infty$ are given for $h/a = 1.5$, 2, 5, and 10. $N_3 = 0$ gives the uncontaminated free surface, with $N_3 \to \infty$ and $\bar{c}_n = -\kappa_n/\tau_n$ in the solid bounding plane configuration. The effect of the presence of the surfactant is also displayed in figure 4, in which log-linear plots of $F(N_3)/F(0)$ against $N_3$ where $F(0)$ is the drag force for a clean surface, are given for $h/a = 1.5$ and $h/a = 2$.

It is of some interest to estimate the accuracy of the asymptotic formulae obtained in I. When $N_3 = O(1)$, the sphere result of I ([59]) is

$$\frac{F_1}{F_\infty} = 1 + \frac{3}{4}\epsilon + \frac{9}{16}(1 + N_3)\epsilon^2 + O(\epsilon^3), \qquad [112]$$

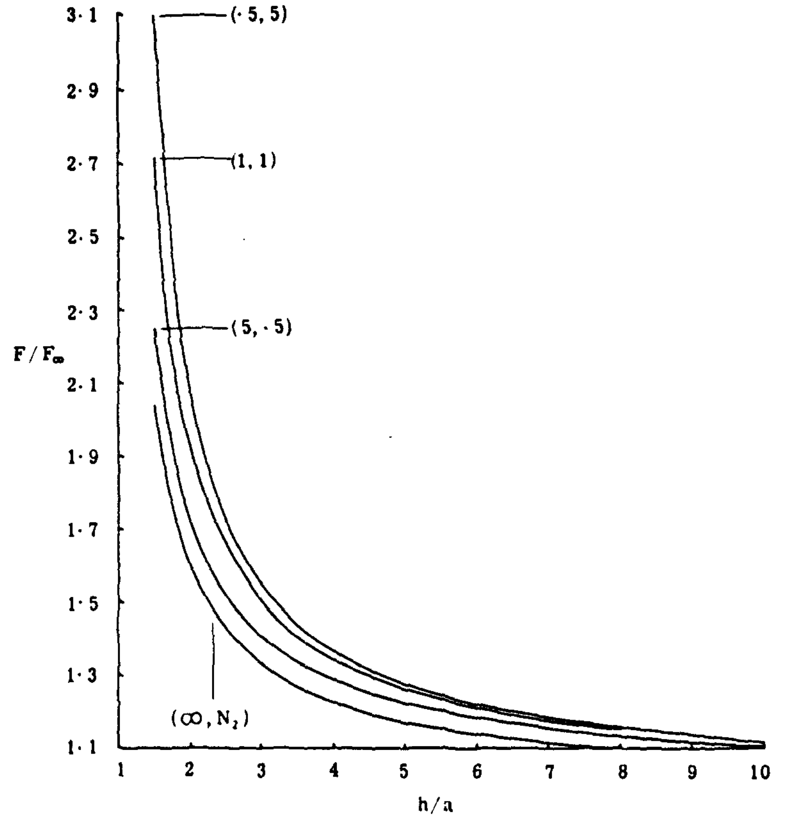


Figure 3. Graphs of $F/F_\infty$ against $h/a$ for the sphere in the surface diffusion case with $(N_1, N_2) = 5, 0.5), (1, 1), (0.5, 5)$ and $(\infty, N_2)$.

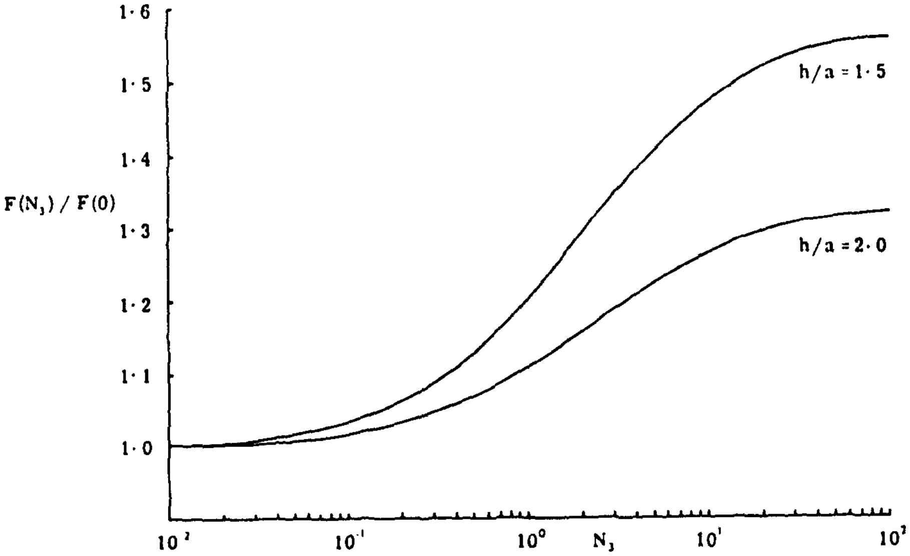


Figure 4. Log-linear plots of $F(N_3)/F(0)$ against $N_3$ for $h/a = 1.5$ and 2 in the adsorbtion/desorbtion case for the sphere.

Table 4. Values of $F/F_\infty$ for $h/a = 1.5$, 2, 5 and 10, and various values of $N_3$

| $N_3$ | h/a = 1.5 | h/a = 2.0 | h/a = 5.0 | h/a = 10.0 |
|---|---|---|---|---|
| 0.0 | 2.0387 | 1.5967 | 1.1756 | 1.0810 |
| 0.025 | 2.0569 | 1.6038 | 1.1763 | 1.0811 |
| 0.05 | 2.0744 | 1.6107 | 1.1771 | 1.0813 |
| 0.075 | 2.0912 | 1.6174 | 1.1778 | 1.0814 |
| 0.1 | 2.1074 | 1.6238 | 1.1785 | 1.0816 |
| 0.25 | 2.1930 | 1.6583 | 1.1824 | 1.0825 |
| 0.5 | 2.3047 | 1.7044 | 1.1882 | 1.0839 |
| 0.75 | 2.3909 | 1.7409 | 1.1933 | 1.0852 |
| 1.0 | 2.4603 | 1.7707 | 1.1977 | 1.0864 |
| 1.25 | 2.5176 | 1.7957 | 1.2017 | 1.0876 |
| 1.5 | 2.5660 | 1.8171 | 1.2052 | 1.0886 |
| 1.75 | 2.6076 | 1.8356 | 1.2085 | 1.0896 |
| 2.0 | 2.6437 | 1.8518 | 1.2114 | 1.0905 |
| 3.0 | 2.7512 | 1.9010 | 1.2209 | 1.0937 |
| 4.0 | 2.8230 | 1.9344 | 1.2281 | 1.0963 |
| 5.0 | 2.8745 | 1.9589 | 1.2337 | 1.0984 |
| 6.0 | 2.9135 | 1.9776 | 1.2382 | 1.1003 |
| 8.0 | 2.9688 | 2.0044 | 1.2451 | 1.1033 |
| 10.0 | 3.0062 | 2.0229 | 1.2502 | 1.1056 |
| 50.0 | 3.1569 | 2.0997 | 1.2748 | 1.1190 |
| 100.0 | 3.1804 | 2.1121 | 1.2796 | 1.1221 |

Table 5. The approximate and computed drag ratios $F_1/F_\infty$ and $F/F_\infty$, for various values of $\epsilon$ and $N_3$

| | | $\epsilon$ = 0.5 | | |
|---|---|---|---|---|
| $N_3$ | 0.5 | 1.0 | 1.5 | 2.0 |
| $F_1/F_\infty$ | 1.5859 | 1.6563 | 1.7266 | 1.7969 |
| $F/F_\infty$ | 1.7044 | 1.7707 | 1.8171 | 1.8518 |
| % error | 6.95 | 6.46 | 4.98 | 2.96 |
| | | $\epsilon$ = 0.2 | | |
| $N_3$ | 0.5 | 1.0 | 1.5 | 2.0 |
| $F_1/F_\infty$ | 1.1838 | 1.1950 | 1.2063 | 1.2175 |
| $F/F_\infty$ | 1.1882 | 1.1977 | 1.2052 | 1.2114 |
| % error | 0.37 | 0.23 | 0.09 | 0.50 |

and in table 5 we compare the approximate drag ratio $F_1/F_\infty$, obtained from [112], with the numerically computed ratio $F/F_\infty$ for $N_3 = 0.5(0.5)2.0$. For $\epsilon = 0.5$ the percentage error ranges from about 7% down to 3%, whereas for $\epsilon = 0.2$ the percentage errors are much smaller. Similar accuracy is obtained for the other models and for the disk.

## 7. CONCLUSION

In the preceding pages the formulation given in I for a solid moving slowly in a semi-infinite liquid, whose surface is contaminated with a surfactant film, has been applied to the particular cases of a translating disk and sphere. All the induced fluid motions have been axisymmetric and quasi-steady, with no restrictions placed on the various parameters in the theory which arise from either the geometry of the configuration or the physics of the film process. Thus the problems result in a considerable amount of numerical analysis, involving the solution of simultaneous Fredholm integral equations of the second kind for the disk, and sets of simultaneous linear equations for the sphere. The paper contains representative numerical results in both tabular and graphical form for the problems investigated, and the opportunity has been taken to vary their mode of presentation so as to avoid repitition and to indicate as clearly as possible the influence of the surfactant on the drag experienced by the solid. The accuracy of the asymptotic drag results in I has also been estimated.

Most detailed attention has been devoted to the film processes of surface diffusion, and absorbtion from and desorbtion to the bulk fluid, but we have included a formulation of the sealed-disk problem in which there is bulk diffusion, at low Peclet number, of a soluble surfactant into the film. No calculations have been made for the bulk diffusion model for a translating spherical particle, but it is clear that the problem is again reducible to the solution of sets of simultaneous linear equations. Thus, for an impervious sphere, the harmonic function $n'$ is represented in bispherical coordinates as

$$n' = (\cosh \xi - s)^{-\frac{1}{2}} \sum_{n=1}^{\infty} \{p_n \cosh (n + \tfrac{1}{2})\xi + q_n \sinh (n + \tfrac{1}{2})\xi\} C_{n+1}^{-\frac{1}{2}}(s). \qquad [113]$$

The condition that $\partial n'/\partial \xi = 0$ on $\xi = \beta$ then leads to an infinite set of equations relating the sequences $\{p_n\}$ and $\{q_n\}$. Equations [97], [99] and [100] still obtain, and condition [10], together with [11], generate sufficient linear equations for the determination of the sequences of constants in [113] and the stream function. The drag is still given by [96].

For simplicity the various film processes have been treated separately, but since the theory is linear it is clearly possible to consider several processes operating simultaneously. A considerable complication in algebra ensues, but no new principles are involved. The approximation of a planar surface throughout the motion has been employed, and although widely used by various authors this is manifestly unsatisfactory for small $a/h$ in the disk case and for $a/h$ near unity for the sphere. Berdan & Leal (1982) and Lee & Leal (1982) have recently studied ways of allowing for the deformation of a surface or interface between two bulk fluids, and it may prove possible to use their ideas in the surfactant case. It is certainly possible to extend the calculations of this paper to the two-fluid case in which a second bulk fluid occupies the region $z < 0$. Again, with no new principles involved, the algebra is more complicated and the number of dimensionless parameters in the analysis increases.

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